

Triple integrals: Same as before, but with 3 variables

$\iiint_R f(x,y,z) dV$ is like the height over a rectangular box

ex: $\iiint_E (xy+z^2) dV$ $E = [0,2] \times [0,1] \times [0,3]$ region is a rectangular prism

$$\begin{aligned} \int_0^2 \int_0^1 \int_0^3 (xy+z^2) dz dy dx &= \int_0^2 \int_0^1 \left[xyz + \frac{z^3}{3} \right]_0^3 dy dx \\ &= \int_0^2 \int_0^1 (3xy+9) dy dx = \int_0^2 \left[\frac{3}{2}xy^2 + 9y \right]_0^1 dx = \int_0^2 \left(\frac{3}{2}x + 9 \right) dx \\ &= \left[\frac{3}{4}x^2 + 9x \right]_0^2 = 3 + 18 = 21 \end{aligned}$$

$$\iiint_R (2x-y) dV \quad R = \{(x,y,z) : \overset{\textcircled{1}}{0 \leq z \leq 2}, \overset{\textcircled{2}}{0 \leq y \leq z}, \overset{\textcircled{3}}{0 \leq x \leq y-z}\}$$

which is in the form: $\textcircled{1} \leq z \leq \textcircled{2}$, $\textcircled{2} \leq y \leq \textcircled{3}$, $h_1(x,z) \leq x \leq h_2(x,z)$

$\textcircled{1}$ is constant, $\textcircled{2}$ is a function of z , $\textcircled{3}$ is a function of y, z

easy to integrate in order $\textcircled{3}, \textcircled{2}, \textcircled{1}$ because is in terms of successively lower amounts of variables

$$\begin{aligned} \int_0^2 \int_0^z \int_0^{y-z} (2x-y) dx dy dz &= \int_0^2 \int_0^z \left[x^2 - xy \right]_0^{y-z} dy dz \\ &= \int_0^2 \int_0^z (z^2 - yz) dy dz = \int_0^2 \left[yz^2 - \frac{zy^2}{2} \right]_0^z dz = \int_0^2 \left(z^4 - \frac{1}{2}z^5 \right) dz \\ &= \left[\frac{z^5}{5} - \frac{z^6}{12} \right]_0^2 = \frac{16}{15} \end{aligned}$$

what if the order was switched?

$$\begin{aligned} \text{if: } \int_0^2 \int_0^{y-z} \int_0^z (2x-y) dy dx dz &= \int_0^2 \int_0^{y-z} \left[2xy - \frac{y^2}{2} \right]_0^z dz \\ &= \int_0^2 \int_0^{y-z} \left(2xz - \frac{z^2}{2} \right) dz dy = \int_0^2 \left[xz^2 - \frac{xz^3}{3} \right]_0^{y-z} dy \end{aligned}$$

$$= \int_0^2 (y-z)^2 z^2 - \frac{1}{3}(y-z)z^3 dz$$

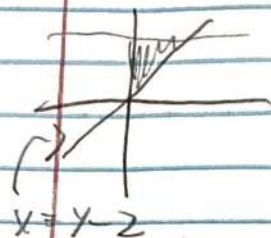
these can't go away unless

reparameterized to make y bounded by x

if bounding y by x : $0 \leq z \leq 2$, $0 \leq y \leq z^2$, $0 \leq x \leq y - z$

end points: $x=0$ $x=y-z \Rightarrow x=z^2-z \Rightarrow 0 \leq x \leq z^2-z$
 $y=0$ $y=z^2$ $x+z \leq y \leq z^2$

$z=0$ $z=2$ $R \{(x,y,z) : 0 \leq z \leq 2,$
 $0 \leq x \leq z^2-z$
 $0 \leq y \leq z^2$



$$\iiint_R (2x-y) dV = \int_0^2 \int_0^{z^2-z} \int_{x+z}^{z^2} (2x-y) dy dx dz$$

$$= \int_0^2 \int_0^{z^2-z} \left[2xy - \frac{y^2}{2} \right]_{x+z}^{z^2} dx dz = \int_0^2 \int_0^{z^2-z} \left(2xz - \frac{z^4}{2} \right) \left(2x(x+z) - \frac{(x+z)^2}{2} \right) dx dz$$

$$= \int_0^2 \int_0^{z^2-z} \left(-\frac{3x^2}{2} + \frac{z^2}{2} + 2xz - \frac{z^4}{2} - xz \right) dx dz = \int_0^2 \left[-\frac{x^3}{2} + \frac{xz^2}{2} - \frac{x^3}{6} - \frac{x^2 z}{2} + \frac{x^2 z^2}{2} \right]_0^{z^2-z} dz$$

$$= \int_0^2 (z^2-z)(z^2-z) \left(-\frac{1}{6}z^2 + \frac{1}{2}z - \frac{1}{6}z^2 \right) dz = \int_0^2 (z^2-z)^2 \left(-\frac{1}{6}z^2 + \frac{1}{2}z - \frac{1}{6}z^2 \right) dz$$

reparameterization is tough, see website

compute volume of tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

$$\text{vol}(T) = \iiint_T 1 dV \quad x+y+z=1$$

$$0 \leq x \leq 1, 0 \leq y \leq 1-x, 0 \leq z \leq 1-x-y$$

$$\text{So } \text{vol}(T) = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 dz dy dx = \int_0^1 \int_0^{1-x} [z]_0^{1-x-y} dy dx$$

$$= \int_0^1 \int_0^{1-x} (1-x-y) dy dx = \int_0^1 \int_0^{1-x} \left[y - xy - \frac{y^2}{2} \right]_0^{1-x} dx$$

$$= \int_0^1 \left((1-x) - x(1-x) - \frac{1}{2}(1-x)^2 \right) dx = \frac{1}{2} \int_0^1 (1-x)^2 dx$$

$$= -\frac{1}{2} \cdot \frac{1}{3} [(1-x)^3]_0^1 = -\frac{1}{6} (-1) = \frac{1}{6}$$